

Optimizing the Design of Space Radiators

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A procedure for optimizing the configuration of a heat pipe/fin radiating element in terms of heat rejected per radiating mass has been developed. The optimization was carried out analytically by expressing the heat radiated per radiating mass in terms of a function involving a dimensionless heat transfer coefficient and the dimensionless thermal gradient at the root of the fin where it joins the heat pipe. The dimensionless Stefan–Boltzmann radiation equation was solved numerically to determine the value that maximizes the function that determines the heat transfer per radiating mass. Once this value is obtained, the optimum width and thickness of the fins as well as the heat radiated per mass can be specified in terms of the operating temperature, emissivity, diameter, and mass/length of the heat pipe, and the density and thermal conductivity of the fin material. The resulting analytical expression can then be used to determine the maximum heat radiated per radiating mass over a wide range of operating conditions, to optimize the design of a specific heat pipe/fin combination, and to conduct analyses of the influence of design and materials properties on the performance of the system. The optimization procedure was carried out for the case of uniformly tapered fins as well as for flat fins.

KEY WORDS: heat pipe/fin radiator; optimization; space radiators.

1. INTRODUCTION

The ability to dissipate large amounts of waste heat is crucial to any high-powered advanced space propulsion system, and the mass and area of the radiating system may well be the limiting design factor for future spacecraft using such a propulsion system. Therefore, it is crucial to optimize the radiator design of such systems to provide the maximum heat

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rejection for the minimum mass. Designers of space heat rejection systems are tending to move away from the conventional "pumped loop" design to a segmented element design consisting of many parallel finned heat pipes [1,2]. The individual element of this system is a heat pipe whose evaporator section is immersed in a manifold or duct containing the fluid to be cooled while the condenser section is equipped with fins that radiate to space. This parallel concept is much less vulnerable to puncture by micrometeoroids and space debris (or to other failure modes) since the loss of a few elements will not significantly affect the total system.

In the design of such a system, careful attention must be given to the width and thickness of the fins. As the heat is radiated from the fin, its local temperature will drop with distance away from the heat pipe making heat rejection less efficient. Thus, if the fin is made too wide, the resulting loss in efficiency will unnecessarily increase the mass of the system. Similarly, the amount of heat that can be transmitted from the heat pipe to the fin will be limited by the thickness of the fin. If the fin is made too thin for its width, the radiating area is not effectively utilized. If the fin is made too thick for its width, the heat radiated will be limited by the width of the fin. In either case, the mass required to radiate the heat will increase unnecessarily. Consideration must also be given to optimizing the ratio of the fin mass to the heat pipe mass. If the mass per length of the heat pipe can be reduced, it is more efficient to also reduce the fin mass per length and either make the heat pipe/fin longer or increase the number of elements in the system in order to radiate a given amount of heat.

The problem can thus be stated as follows: given the mass per unit length of the heat pipe that feeds the fins and its operating temperature, determine the optimum fin width and thickness as a function of thermal properties that maximizes the heat radiated per overall radiating mass (heat pipe plus fin). The length of the heat pipe/fin combination will then be determined by the thermal power the heat pipe is capable of delivering.

The heat radiated per element mass was shown to be directly proportional to a dimensionless parameter involving the thermal gradient at the fin root as well as a heat transfer coefficient containing the material properties and operating temperature. The heat transfer coefficient that maximizes this dimensionless parameter was obtained by solving a dimensionless form of the Stefan-Boltzmann radiation equation numerically. Once the optimum value of this heat transfer coefficient was determined numerically, exact analytical expressions are obtained for the maximum heat radiated per element mass in terms of geometry, operating temperature, and physical properties for the optimal fin mass to the heat pipe mass and for the optimal fin width and thickness.

The resulting exact analytical expressions can then be used to determine the maximum heat radiated per radiating mass over a wide range of operating conditions to optimize the design of a specific heat pipe/fin combination and to conduct analyses of the influence of design and materials properties on the performance of the system without having to perform additional numerical computations. Although numerical computations allow more detail, such as the effect of the temperature dependence of the thermal conductivity and emissivity to be explored, analytical solutions provide more insight into the interplay of the various parameters and are more efficient for optimization and sensitivity analyses.

The optimization procedure was carried out for the case of uniformly tapered fins as well as for flat fins.

2. ANALYSIS

The following analysis assumes the fin to be L meters long, w meters wide, and D meters thick with a conductivity K ($\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}$) as shown in Fig. 1. The edge $x = 0$ along the length L is held at constant temperature T_0 by a heat pipe or other source while both surfaces of the fin are allowed to radiate into free space ($T = 0$). The fin is considered thin enough so that there are no gradients across its thickness and no significant radiation from the edges, and radiation from both surfaces is assumed.

Assuming $D \ll w$, the problem can be analyzed using one-dimensional heat flow to obtain the temperature distribution over the width of the fin. The governing equation is

$$KD \frac{\partial^2 T}{\partial x^2} = 2\sigma \varepsilon T(x)^4, \tag{1}$$

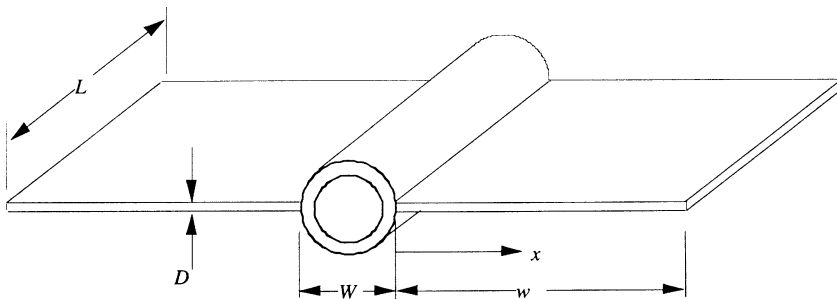


Fig. 1. Schematic of a heat pipe/fin radiating element.

where K is the thermal conductivity of the fin material, σ is the Stefan-Boltzmann constant, and ε is the emissivity of the fin surface. Introducing dimensionless terms, $\Theta = T/T_0$ and $\xi = x/w$, this equation becomes

$$\frac{\partial^2 \Theta}{\partial \xi^2} = \frac{2\sigma \varepsilon w^2 T_0^3}{KD} \Theta(\xi)^4 = \beta^2 \Theta(\xi)^4 \quad (2)$$

where the dimensionless heat transfer coefficient, β , is defined as

$$\beta^2 = \frac{2\sigma \varepsilon T_0^3 w^2}{KD} \quad (3)$$

and the boundary conditions are $\Theta(0) = 1$ and $\Theta'(1) = 0$. While both K and ε may vary with temperature, the variation is generally small over the limited range over which the fin temperature will be allowed to vary; hence, the use of average values for these properties is justified.

The total heat per length radiated from a heat pipe/fin element is twice the heat conducted into an individual fin plus the radiation from the upper and lower surfaces of the heat pipe itself, or

$$Q/L = -2KD \left. \frac{\partial T}{\partial x} \right|_{x=0} + 2W\varepsilon\sigma T_0^4 = -2 \frac{KDT_0}{w} \Theta'(0) + 2W\varepsilon\sigma T_0^4, \quad (4)$$

where W is the effective radiating width of the heat pipe. Let μ be the linear density of the radiating portion of the heat pipe. The mass of the radiating system is $M = L(\mu + 2\rho wD)$. Thus the amount of heat radiated per total radiating mass is

$$Q/M = \frac{-2K(D/w)T_0\Theta'(0) + 2W\varepsilon\sigma T_0^4}{(\mu + 2\rho wD)}. \quad (5)$$

The task now is to decide how to apportion the w and D for a given μ to maximize the total heat radiated per radiating system mass. To perform this optimization, we introduce another dimensionless parameter R , the ratio of fin mass to the mass of the radiating portion of the heat pipe,

$$R = \frac{2\rho wD}{\mu} \quad (6)$$

and use the definition of β (Eq. (3)) with the above definition of R to express the thickness D and width w in terms of β and R ;

$$D = \left(\frac{\mu R}{2\rho\beta} \right)^{2/3} \left(\frac{2\sigma \varepsilon T_0^3}{K} \right)^{1/3}, \quad (7)$$

and

$$w = \beta \left(\frac{\mu R}{2\rho\beta} \right)^{1/3} \left(\frac{K}{2\sigma\epsilon T_0^3} \right)^{1/3} \tag{8}$$

Using these expressions in Eq. (5),

$$\frac{Q}{M} = \frac{2\sigma\epsilon T_0^4}{\mu} \left(\left(\frac{2\mu K}{\rho\sigma\epsilon T_0^3} \right)^{1/3} \frac{R^{1/3}}{(1+R)} \frac{|\Theta'(0)|}{\beta^{4/3}} + \frac{W}{(1+R)} \right) \tag{9}$$

We can now optimize the system with respect to β and R . Maximizing the function $|\Theta'(0)|/\beta^{4/3}$ will maximize Q/M independent of R . Taking a partial derivative with respect to R and setting it equal to zero, we obtain a nonlinear equation,

$$(2R - 1) = - \frac{3\beta^{4/3}}{|\Theta'(0)|} \left(\frac{\rho\sigma\epsilon T_0^3}{2\mu K} \right)^{1/3} W R^{2/3} \tag{10}$$

that must be solved for R . Note that if radiation from the heat pipe is negligible, $W=0$ and $R=0.5$ and if $W > 0$, $R < 0.5$. This implies that the mass of the fin must always be less than 1/3 the mass of the radiating system, which places a premium on reducing the mass per length of the heat pipe.

The governing equation, Eq. (2), can be solved by a second order Runge–Kutta integration if the initial conditions $\Theta(0)$ and $\Theta'(0)$ are known. However, in our case, only $\Theta(0)$ is known and we must enforce $\Theta'(1) = 0$. This condition can be met using a Newton–Raphson iterative process with the approximate solution (see Appendix A) as a starting point.

Recall from Eq. (9) that the Q/M can be maximized with respect to β independently of R by maximizing the function $|\Theta'(0)|/\beta^{4/3}$. This function is computed from the numerical solution of Eq. (2) for selected values of β , again using the approximate solution as a guide. The results are shown in Fig. 2. The resulting points were fitted by a third-degree polynomial, and the value $\beta = 0.9190$ that maximizes the function was determined by setting the derivative of the polynomial to zero and solving for β . The corresponding $\Theta'(0) = -0.4778$ and $|\Theta'(0)|/\beta^{4/3} = 0.5348$ were then computed. Once these values have been determined for the dimensionless form of the governing equation, they apply to the optimized case for any set of dimensional variables.

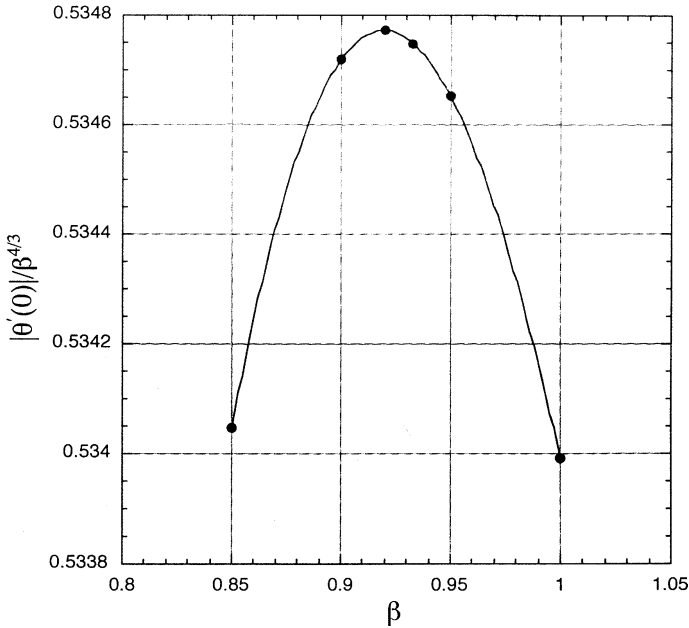


Fig. 2. Plot of the function $|\Theta'(0)|/\beta^{4/3}$. Value of β that maximizes the function is found to be 0.9190.

Inserting the optimal values for β and $\Theta'(0)$ into Eq. (10),

$$(2R - 1) = -\frac{3\beta^{4/3}}{|\Theta'(0)|} \left(\frac{\rho\sigma\epsilon T_0^3}{2\mu K} \right)^{1/3} W R^{2/3} = -5.609 \left(\frac{\rho\sigma\epsilon T_0^3}{2\mu K} \right)^{1/3} W R^{2/3}. \tag{11}$$

After solving Eq. (11) for the optimal value for R , the optimum width and thickness for the fin is obtained from Eqs. (7) and (8) and the Q/M may be found from

$$\frac{Q}{M} = \frac{2\sigma\epsilon T_0^4}{\mu(1+R)} \left(\left(\frac{2\mu K R}{\rho\sigma\epsilon T_0^3} \right)^{1/3} 0.5348 + W \right). \tag{12}$$

This procedure will produce the optimum design conditions and the value for Q/M over the entire range of operating temperatures and other properties with no approximations involved. It also provides an analytical tool for assessing how performance depends on the conductivity of the fin material, the emissivity of the radiating surfaces, and the linear density of the heat pipe.

3. EFFECT OF TAPERING THE FINS

A fairly obvious way of increasing the radiated heat without increasing the system mass would be to increase the thickness of the fin at the root and taper the thickness uniformly to zero at the tip. This configuration would allow more heat to be conducted into the fin in the region of highest temperature in order to counter the rapid drop in temperature over the width of the fin. If the fin is tapered from D at the root to 0 at $x = w$, the heat flow equation may be written as

$$KD \frac{d}{dx} \left[(1 - x/w) \frac{dT}{dx} \right] - 2\sigma\epsilon T^4 = 0 \tag{13}$$

or in dimensionless parameters,

$$\frac{d}{d\xi} \left[(1 - \xi) \frac{d\Theta}{d\xi} \right] = \frac{2\sigma\epsilon w^2 T_0^3}{KD} = \beta^2 \Theta. \tag{14}$$

With a tapered fin, the total fin mass per length is $\rho w D$ and the ratio of fin mass to heat pipe mass is now given by $R = \rho D w / \mu$. Using this and the definition for β , the values for D and w are now given by

$$D = \left(\frac{\mu R}{\rho \beta} \right)^{2/3} \left(\frac{2\sigma\epsilon T_0^3}{K} \right)^{1/3}, \tag{15}$$

$$w = \beta \left(\frac{\mu R}{\rho \beta} \right)^{1/3} \left(\frac{K}{2\sigma\epsilon T_0^3} \right)^{1/3}. \tag{16}$$

These equations are similar to Eqs. (7) and (8), differing only by the absence of a factor of two in the denominator in the first term. Including the heat radiated from the heat pipe, the Q/M can be written in the form of Eq. (9),

$$\frac{Q}{M} = \frac{2\sigma\epsilon T_0^4}{\mu} \left(\left(\frac{4\mu K}{\rho\sigma\epsilon T_0^3} \right)^{1/3} \frac{R^{1/3}}{(1+R)} \frac{|\Theta'(0)|}{\beta^{4/3}} + \frac{W}{(1+R)} \right). \tag{17}$$

The only difference between this and Eq. (9) is the factor $4^{1/3}$ instead of $2^{1/3}$ in the numerator. Again Q/M can be maximized with respect to β independent of R by solving Eq. (14) numerically to obtain $\Theta'(0)$ that conforms to the boundary conditions $\Theta(0) = 1$ and $\Theta'(1) = 0$ for various values of β . The function $|\Theta'(0)|/\beta^{4/3}$ can then be evaluated as a function of β as before. The maximum absolute value of this function, 0.4848,

is obtained for $\beta = 0.8825$ and $\Theta'(0) = -0.4104$. The optimal value for R may be obtained as before by maximizing Eq. (17) with respect to R ;

$$(2R - 1) = -\frac{3\beta^{4/3}}{|\Theta'(0)|} \left(\frac{\rho\sigma\epsilon T_0^3}{4\mu K} \right)^{1/3} W R^{2/3} = -6.188 \left(\frac{\rho\sigma\epsilon T_0^3}{4\mu K} \right)^{1/3} W R^{2/3}. \quad (18)$$

After finding the optimal value for R from Eq. (18) the maximum Q/M can be found from

$$\begin{aligned} \frac{Q}{M} &= \frac{2\sigma\epsilon T_0^4}{\mu(1+R)} \left(\left(\frac{4\mu K R}{\rho\sigma\epsilon T_0^3} \right)^{1/3} 0.4848 + W \right). \\ &= \frac{2\sigma\epsilon T_0^4}{\mu(1+R)} \left(\left(\frac{2\mu K R}{\rho\sigma\epsilon T_0^3} \right)^{1/3} 0.6110 + W \right). \end{aligned} \quad (19)$$

Comparing this result with Eq. (12), tapering the fins can produce up to a 14% gain in heat rejected per radiating mass, depending on the value of W .

4. PERFORMANCE COMPARISONS

We now utilize this above optimization technique to compare radiator performance of different materials at different temperatures. We compare fins made from Al, Be, and an advanced C-C composite (K1100) and compare the optimized configuration against the heat pipe/fin configuration described in Ref. 2. In all cases the heat pipe was considered to be a 91 cm long, 2.5 cm diameter thin-walled Nb-Zr alloy tube surrounded by 1 mm of the fin material. The effective radiating width W was taken to be the diameter rather than half the circumference of the heat pipe since it will exchange some radiation with the fins. The emissivity was taken to be 0.85. This and other thermophysical properties quoted in Table I are meant to be representative rather than precise values. The results are tabulated in Table I.

Several interesting aspects of the optimization procedure and the effects of materials properties can be seen in these calculations. First note the dramatic increase in performance obtained by using the higher-conductivity, lower-density highly ordered pyrolytic graphite fin material [3] compared with Al or Be. Also note the 20% improvement in Q/M obtained by optimizing the fin configuration of this system even though the un-optimized configuration actually radiates 12% more heat. This

Table I. Performance Comparisons (Values are Derived Except as Noted in Footnotes)

Fin Material Configuration	Al, optimized	Be, optimized	K1100, Ref. 2	K1100, optimized	K1100, optimized	K1100, tapered
T_0 (K) ^a	700	700	700	700	1000	1000
ρ (kg·m ⁻³) ^{b,c}	2700	1800	1218	1218	1218	1218
ε (-) ^a	0.85	0.85	0.85	0.85	0.85	0.85
K (W·m ⁻¹ ·K ⁻¹) ^{b,c}	257	200	1000	1000	1000	1000
L (m) ^a	0.914	0.914	0.914	0.914	0.914	0.914
W (cm) ^a	2.50	2.50	2.50	2.50	2.50	2.50
μ (kg·m ⁻¹)	0.334	0.263	0.217	0.217	0.217	0.217
β (-)	0.919	0.919	0.431	0.919	0.919	0.882
R (-)	0.300	0.296	0.841	0.370	0.328	0.345
$\Theta'(0)$ (-)	-0.478	-0.478	-0.153	-0.478	-0.478	-0.410
w (mm)	49.52	47.96	75.0	94.4	63.5	79.2
D (mm)	0.374	0.45	1.00	0.349	0.461	0.776
M (kg)	0.396	0.311	0.365	0.272	0.254	0.267
$\Theta(1)$ (-)	0.798	0.798	0.928	0.798	0.798	0.715
η (-)	0.56	0.56	0.821	0.56	0.56	0.53
Q (W)	1715	1677	3134	2791	8538	9559
Q/M (W·kg ⁻¹)	4328	5388	8576	10260	32390	35750
M/A (kg·m ⁻²)	3.494	2.817	2.285	1.393	1.897	1.835

^a Assumed values.

^b Tables in Ref. 5.

^c Properties of highly ordered pyrolytic graphite from Ref. 3.

improvement in Q/M comes about from reducing the fin mass by 34% by making the fin much thinner and somewhat longer.

The $T_0 = 1000$ K cases were run to illustrate how the performance improves at higher temperatures as well as the effect of tapering the fins. First note that there is only approximately a threefold increase in Q/M when the operating temperature is increased from 700 to 1000 K instead of the fourfold increase expected from the T^4 law. This comes about because of the T^2 dependence of the optimized D/w ratio in Eq. (5), which gives the fin portion of the radiation a T^3 dependence. It can also be seen that tapering the fin results in a 10% increase in the Q/M for this configuration.

The quantity η in the table is the fin efficiency. It is defined as the heat actually radiated from the fin divided by the heat that would be radiated if the entire fin were held at T_0 . It is determined by dividing the fin contribution to Eq. (4), $-2K(D/w)T_0\Theta'(0)$ by $2\sigma\varepsilon wT_0^4$. The Ref. 2 configuration has a much higher fin efficiency than the optimum configuration because the shorter, fatter fins maintain a much higher average

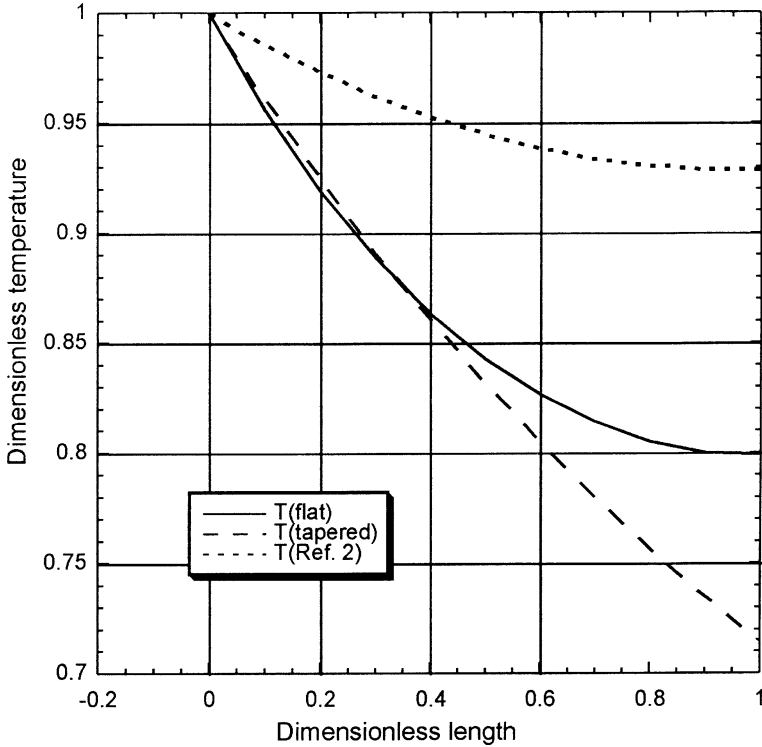


Fig. 3. Dimensionless temperature (T/T_0) as a function of fractional distance along the fin width for different fin configurations.

temperate as can be seen in Fig. 3, but at the expense of added mass. However, as shown in the Appendix A, the highest fin efficiency is attained by shrinking β to 0, which does not produce any radiated heat. Therefore, fin efficiency is not a particularly useful indicator of performance.

The Ref. 2 configuration maintains a higher average temperature because of its shorter fatter fins. The tapered fin suffers the greatest temperature drop but radiates more heat because of its greater length. The thermal gradient $\Theta'(0)$ for the tapered fin is slightly less than that for the flat fin, but more heat is being transferred from the heat pipe to the fin because the tapered fin is almost twice as thick at the root.

The cases considered all assumed a length of 0.91 m for the radiating system. The Q/M should be independent of this length so long as the heat pipe can deliver the heat required to maintain the root temperature at the specified T_0 .

5. SUMMARY

A procedure for optimizing the configuration of a heat pipe/fin radiating element in terms of heat rejected per unit mass has been developed using a combination analytical/numerical procedure that provides analytical expressions for the maximum heat radiated per radiating mass as well as for the optimum fin width and thickness. The expressions developed for both flat and tapered fins are valid over any set of operation conditions and involve no approximations. It was shown that the Q/M is inversely proportional to the linear density of the heat pipe, which places a premium on the development of lightweight heat pipes. The heat radiated from the fin is shown to be proportional to the cube of the heat pipe temperature, to the $2/3$ power of the emissivity, and to the $1/3$ power of the ratio of thermal conductivity to density of the fin material. Tapering the fins can provide up to 14% greater radiation per mass, depending on the effective width of the radiating heat pipe. Increasing the effective radiating width W of the heat pipe itself by fabricating it with an elliptical cross section can also increase the Q/M .

The optimization procedure is relatively insensitive to the dimensionless heat transfer coefficient β , and thus the thermophysical properties than it involves as may be seen from Fig. 2. A variation of $\pm 5\%$ in β reduces the optimum Q/M by only 0.56%, which is further justification for using average values for K and σ , especially when the temperature across the fin varies only by $\sim 25\%$ of its maximum value as may be seen in Fig. 3.

The analysis only included the radiating mass of the heat pipe/fin combination and did not include the evaporator portion of the heat pipe or the mass of the duct system that carries heat to the heat pipe. Also, the analysis assumed the mass of the radiating system and the heat radiated increase linearly with the length of the element. Clearly the ability of the heat pipe to carry the required heat without a drop in temperature, along with the mass of the duct and evaporator, must be considered in determining the optimum length of the radiating elements and number of elements needed to radiate the required amount of heat.

APPENDIX A. APPROXIMATE SOLUTIONS TO THE STEFAN-BOLTZMANN RADIATION EQUATION

The governing equation for the flat fin case, Eq. (2) can be expanded about $\Theta = 1$ to give

$$\frac{\partial^2 \Theta}{\partial \xi^2} = \beta^2 [1 - 4(1 - \Theta(\xi))] \approx \beta^2 [4\Theta(\xi) - 3]. \quad (\text{A.1})$$

The solution [4] is

$$\Theta = \frac{3}{4} + \frac{\cosh[2\beta(1-\xi)]}{4 \cosh[2\beta]} \quad (\text{A.2})$$

and

$$\Theta'(0) = -\beta \frac{\tanh(2\beta)}{2} \quad (\text{A.3})$$

The heat per length flowing into the fin at $x=0$ is given by

$$Q_{cond}/L = -KD \left. \frac{\partial T}{\partial x} \right|_0 \approx \frac{KDT_0}{2w} \beta \tanh[2\beta] = 2\sigma \varepsilon w T_0^4 \frac{\tanh[2\beta]}{2\beta} \quad (\text{A.4})$$

Dividing by the heat radiated from both sides of a fin whose width is w , the fin efficiency is $\eta = \frac{\tanh[2\beta]}{2\beta}$ which is maximized in the limit $\beta \rightarrow 0$. Inserting Eq. (A.3) into Eq. (9),

$$\frac{Q}{M} \approx \frac{2\sigma \varepsilon T_0^4}{\mu(1+R)} \left[\left(\frac{\mu K}{2\rho \sigma \varepsilon T_0^3} \right)^{1/3} \frac{R^{1/3} \tanh(2\beta)}{(2\beta)^{1/3}} + W \right]. \quad (\text{A.5})$$

The quantity $\tanh[2\beta]/(2\beta)^{1/3}$ must be maximized with respect to β which requires $\beta=0.709$. This value, along with Eq. (A.3), provided the starting conditions for the iterative numerical solutions.

The solution to the approximate heat flow equation for the tapered fin case is [3]

$$\Theta \approx \frac{3}{4} + \frac{I_0[4\beta(1-\xi)^{1/2}]}{4I_0[4\beta]} \quad (\text{A.6})$$

where I_0 is the modified Bessel function of the First Kind of order zero. The derivative is

$$\left. \frac{\partial \Theta}{\partial \xi} \right|_0 \approx -\frac{I_1[4\beta] \beta}{I_0[4\beta] 2}. \quad (\text{A.7})$$

Inserting this derivative into Eq. (17), the quantity to be maximized is $I_1[4\beta]/\beta^{1/3} I_0[4\beta]$, which occurs for $\beta=0.654$. As before, this value, along with Eq. (A.7), provides the starting points for the iterative numerical solution for the tapered fin.

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